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THE CRITICAL EXTERNAL PRESSURE OF CYLINDRICAL TUBES

BY R. VON MISES

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TRANSLATED AND ANNOTATED
BY D. F. WINDENBURG

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A circular, cylindrical boiler flue in the pressure chamber of a steam boiler is loaded similarly to a vertically, centrally loaded column: as long as the pressure remains under a certain limit we have stable equilibrium and a uniform contraction of the material on all sides. For greater pressures, we have a buckling or bulging of the tube. (See C. Bach: "Elastizität und Festigkeit", 1905 p.273).

Observations of the critical pressure, corresponding to the great practical significance of the question, have already been abundantly made. The corresponding theory, quite similar to the Euler buckling theory, was developed first for the tube of infinite length only. Here, the instability pressure is given by

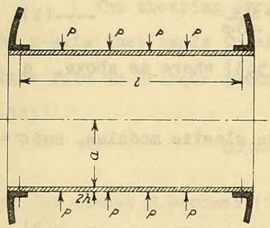

Fig. 1. Cylindrical tube under external pressure p.

$$p = \frac{2E}{1-\sigma^2}\left(\frac{h}{a}\right)^3 \qquad \text{---------- (a)}$$

where "a" designates the inner radius, 2h the shell thickness of the tube (Fig. 1), E the modulus of elasticity and σ Poisson's ratio. This equation was first developed by Bresse in 1829. (See Love-Timpe "Lehrbuch der Elastizität", Leipzig 1907, p. 637). The complete expression for tubes of finite length was treated by R. Lorenz ("Physikal. Zeitschrift", 1911, p. 257) who in order to avoid the difficulties of computation introduced a series of neglected terms. R.V. Southwell (Phil.Mag., Vol.25, 1913, p.687) has also recently given an approximate solution.

The following development is from the rigorous theory of thin elastic shells andderives the accurate expression for the critical pressure. For practical computations, the results can be easily simplified according to the circum-

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stances. In most cases, the annexed table will be sufficient. The values given here differ as much as 20% or more from the Lorenz values and in the direction of a better agreement with observation. The essential characteristic of the buckling phenomena, the unlimited increase of the number of lobes as the length of the tube decreases, has been previously emphasized in the investigations mentioned above.

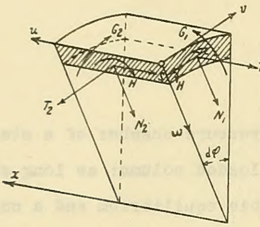


Fig. 2. Stresses and displacements in a volume element.

The extension of the investigation to the region above the elastic limit, and the possibility of a theory for the corrugated tube, will be discussed in a section at the close of this article.

1. The Elastic Equations.

Let a point of the cylindrical surface (See Fig. 2) have the coordinates x, measured in the direction of the axis, and φ, measured on the circumference of the cross section. Let the elastic displacement of this point be u in the direction of the axis, v in the direction of the tangent to the circle, and w in the direction of the radius, measured inward. We then express the strains ε_1, and ε_2 for the u- and v- directions and the angular change γ in the u-v plane as follows:

$$\varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \frac{1}{a}\left(\frac{\partial v}{\partial \varphi} - w\right), \quad \gamma = \frac{\partial v}{\partial x} + \frac{\partial u}{a\partial \varphi} \qquad \text{----(1)}$$

(See Love, The Mathematical Theory of Elasticity, p.543) where as above, a denotes the radius of the cylinder.

Let 2h designate the shell thickness, E the elastic modulus, and $\sigma = 0.3$, Poisson's ratio. With the simplifying substitution

$$c = \frac{2\,Eh}{1 - \sigma^2} \qquad \text{----------------------------(2)}$$

we can easily find the values for the various longitudinal and transverse forces, namely, the normal forces T_1 and T_2 and the shear force S, Fig.2, as follows: (Translator's note: These forces are really stresses multiplied by the thickness).

$$T_1 = c(\varepsilon_1 + \sigma\varepsilon_2), \quad T_2 = c(\varepsilon_2 + \sigma\varepsilon_1), \quad S = \frac{c}{2}(1 - \sigma)\gamma \qquad \text{------(3)}$$

Since the shear modulus is $\frac{E}{2(1+\sigma)}$, the solution of the first two equations of (3) for ε_1, and ε_2 yields the well known simple relation

$$\varepsilon_1 = \frac{1}{E}\,\frac{T_1 - \sigma T_2}{2h}, \qquad \varepsilon_2 = \frac{1}{E}\,\frac{T_2 - \sigma T_1}{2h}$$

(Translator's note: In the text the shear modulus is given $\frac{E}{2}(1 + \sigma)$).

The shear forces N_1 and N_2 that act in the w - direction cannot be expressed by means of displacements, but can be determined only through bending equilibrium.

From the previously considered resultant stresses, three different stress moments, G_1, G_2, and H, arise. The moment G_1 is due to the normal stresses in the cross-section of which T_1 was the resultant. It causes in the first line a curvature K_1 of the cylinder-generator. Similarly, the normal stresses in the longitudinal cross section give a bending moment G_2, that changes the curvature $\frac{1}{a}$ of the line of intersection by an amount K_2. We have then (See Love, p.543)

$$K_1=\frac{\partial^2 w}{\partial x^2}, \quad K_2=\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2}+\frac{\partial v}{\partial \varphi}\right) \qquad \text{------(4)}$$

(Translator's note: In the text $\frac{1}{a}$ is given in the place of $\frac{1}{a^2}$) and considering the cross contraction (Poisson's ratio) similar to eq.(3) (See Love, p.530, eq.37)

$$G_1 = -c\,\frac{h^2}{3}\,(K_1+\sigma K_2), \quad G_2 = -c\,\frac{h^2}{3}\,(K_2+\sigma K_1) \qquad \text{------(5)}$$

The factor $\frac{h^2}{3}$ in eq. (5) together with the 2h contained in c, eq. (2), gives the moment of inertia of the rectangular cross-section of height 2h and breadth unity.

The shearing stresses whose resultant is N_1, produce a moment H about the x-axis that tends to make the rectangular element $dx\,a\,d\varphi$ of the tangential plane into an oblique quadrilateral. The deformation in this sense is (See Love, p.543)

$$\omega=\frac{1}{a}\,\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \varphi}+v\right) \qquad \text{------(6)}$$

and the moment H becomes (See Love, p.530, eq.37)

$$H = c\,\frac{h^2}{3}\,(1-\sigma)\,\omega \qquad \text{----------------(7)}$$

(Translator's note: In the text h^3 is given in place of h^2).

An equally large turning moment about the v-direction acts in the longitudinal section.

2. The Conditions of Equilibrium.

The equilibrium of the forces in the u- and v-directions requires (See Love, p.535, eq.(45))

$$\frac{\partial T_1}{\partial x}+\frac{\partial S}{a\,\partial \varphi}=0, \quad \frac{\partial T_2}{a\,\partial \varphi}+\frac{\partial S}{\partial x}+\frac{N_2}{a}=0 \qquad \text{----------(8)}$$

(Translator's note: See also Prescott's "Applied Elasticity", p.549, eq.17.98, and p.550, eq. 17.102).

The last term in the second equation is based on the fact that the two longitudinal sections bordering the element are not parallel.

In order to be able to set up the equations for the w - direction, we must know the value of the radius of curvature after the deformation. This is, however, $\rho = a - w - \frac{\partial^2 w}{\partial \varphi^2}$, and since (conditions of equilibrium in the w - direction), (Translator's note: See Prescott's "Applied Elasticity", p.549, equation 17.101), $p - \frac{\partial N_1}{\partial x} - \frac{\partial N_2}{\rho \partial \varphi} + \frac{T_2}{\rho} = 0$, must hold, it follows, by the omission of quantities of higher order:

$$\frac{T_2}{a} - \frac{\partial N_1}{\partial x} - \frac{\partial N_2}{a \partial \varphi} = -\frac{p}{a}\left(a - w - \frac{\partial^2 w}{\partial \varphi^2}\right) \qquad \text{-------(9)}$$

The two unknown shearing forces N_1 and N_2 are determined with the help of the equations showing that the summation of the moments about the u- and v-directions are equal to zero. (See Love, p.536, eq.46)

$$\frac{\partial H}{\partial x} - \frac{1}{a}\frac{\partial G_2}{\partial \varphi} - N_2 = 0\,, \quad \frac{\partial H}{a \partial \varphi} - \frac{\partial G_1}{\partial x} - N_1 = 0 \qquad \text{------(10)}$$

Equations (8), (9), and (10), in combination with the elastic relations of the previous section, give the complete statement for every problem dealing with cylindrical shells. A particular solution corresponding to the symmetrical compression through the external pressure p is obtained with u = v = 0. It follows then that $S = N_1 = N_2 = G_1 = G_2 = H = 0$, and from eq. (9): $T_2 = -ap$. Further eq. (3) gives $T_1 = \sigma T_2$ and eq. (1) $\mathcal{E}_2 = -\frac{ap}{c} = -\frac{w}{a}$, $w = \frac{a^2 p}{c}$. (Translator's note: The results are easily obtained since w is independent of x and φ).

If we subtract this value of w from the actual displacement u, v, w, then the remaining part satisfies the same conditions, provided that the first term in the parenthesis in eq. (9) is dropped. (Translator's note: This means that the uniform radial compressive displacement is neglected in comparison with displacements due to buckling. That is, the displacements due to buckling are measured from the original position of the neutral axis of the shell as is shown in Figs. 4 to 6. Subtracting $\frac{a^2 p}{c}$ from w in (1) and substituting the resulting value of T_2 obtained from (3) in (9) gives equation (9') directly). In place of equation (9) we have, therefore,

$$\frac{T_2}{a} - \frac{\partial N_1}{\partial x} - \frac{\partial N_2}{a \partial \varphi} = \frac{p}{a}\left(w + \frac{\partial^2 w}{\partial \varphi^2}\right) \qquad \text{--------(9')}$$

We can now, through the elimination of N_1 and N_2, derive the following three equilibrium equations from equations (8), (9'), and (10):

$$a^2 \frac{\partial^2 T_1}{\partial x} + a\,\frac{\partial^2 S}{\partial \varphi \partial x} = 0 \qquad \text{--------(I)}$$

$$-a^2\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_2}{\partial\varphi^2}+\frac{\partial^2 H}{\partial\varphi\partial x}-\frac{\partial^2 G_2}{a\partial\varphi^2}=0 \qquad \text{---------(II)}$$

$$a\frac{\partial^2 G_1}{\partial x^2}-\frac{\partial^2 H}{\partial x\partial\varphi}-a^2\frac{\partial^2 T_1}{\partial x^2}+\frac{\partial^2 T_2}{\partial\varphi^2}+T_2=p\left(w+\frac{\partial^2 w}{\partial\varphi^2}\right) \qquad \text{--------(III)}$$

If the forces and moments are expressed through the deformations, and these through the displacements u, v, and w, we have in equations I to III the necessary equations for the determination of u, v, and w.

3. Expression for Displacements.

Since the equations I to III become linear and homogeneous after the introduction of u, v, and w, and do not contain the coordinates x, φ explicitly, we can substitue sine and cosine terms for the dependent variables, u, v, w in a well known manner. Let us write

$$u = A\sin n\varphi\sin\frac{\alpha x}{a} \qquad v = B\cos n\varphi\cos\frac{\alpha x}{a}$$
$$w = \sin n\varphi\cos\frac{\alpha x}{a} \qquad \text{------(11)}$$

The change of sine and cosine is governed by the equations themselves as we will see below. The term n can be used to signify a real integral number, and the term α a quantity such that $\frac{\alpha l}{a\pi}$ makes an uneven integral number, where l is the free length of the tube. For only if this expression is an uneven integer are the radial displacements w for $x = \pm\frac{l}{2}$ equal to zero. It is sufficient for us to consider the case in which the uneven integral number has the value 1 (one), since all other cases (multiple buckling) are then easily explained. We have then

$$\alpha=\frac{a\pi}{l} \qquad \text{---------------(12)}$$

If we substitute equation (11) in equations (1), (4), and (6), we get

$$\varepsilon_1=\frac{\alpha A}{a}w,\quad \varepsilon_2=-\frac{nB+1}{a}w,\quad \gamma=\frac{nA-\alpha B}{a}\cos n\varphi\sin\frac{\alpha x}{a}$$
$$K_1=-\frac{\alpha^2}{a^2}w,\quad K_2=-\frac{n(n+B)}{a^2}w$$
$$\omega=\frac{-\alpha(n+B)}{a^2}\cos n\varphi\sin\frac{\alpha x}{a}$$

Let us introduce these values in equations (3), (5), and (7). For simplification we will write $(1-\sigma)(nB+1) = C$, $nB + 1 - \alpha A = D$.

Then

$$T_1=-\frac{c}{a}(D-C)w,\quad T_2=-\frac{c}{a}(\sigma D+C)w$$

$$a\frac{\partial^2 S}{\partial\varphi\partial x}=\frac{c}{2a}\left[(1-\sigma)(n^2D-\alpha^2)-C(n^2-\alpha^2)\right]w \qquad \text{---------(13)}$$
$$G_1=\frac{h^2c}{3a^2}\left[\alpha^2+\sigma\left(n^2-1+\frac{C}{1-\sigma}\right)\right]w$$

$$G_2 = \frac{h^2 c}{3a^2}\left[\sigma\alpha^2 + \left(n^2 - 1 + \frac{c}{1-\sigma}\right)\right] w \qquad \text{-----(13) (Cont'd.)}$$

$$a\frac{\partial^2 H}{\partial x \partial \varphi} = \frac{h^2 c}{3a^2}\left[\alpha^2(1-\sigma)(n^2-1) + \alpha^2 C\right] w$$

The substitution of the expressions (13) in Eqs. (I) to (III) with the simplifications

$$\frac{h^2}{3a^2} = x, \; p\frac{a}{c} = p\frac{a}{2h}\,\frac{1-\sigma^2}{E} = y \qquad \text{------------(14)}$$

gives the following three linear equations in C and D:

$$D\left[2\alpha^2 + (1-\sigma)n^2\right] - C\left[\alpha^2 + n^2\right] = \alpha^2(1-\sigma) \qquad \text{--------(I')}$$

$$D\left[\sigma n^2 - \alpha^2\right] + C\left[\alpha^2 + n^2 + a_1 x\right] = a_2 x \qquad \text{--------(II')}$$

$$D\left[\sigma n^2 - \alpha^2 - \sigma\right] + C\left[n^2 + \alpha^2 - 1 + a_3 x\right] = y(1-n^2) + a_4 x \qquad \text{-------(III')}$$

Here the coefficients of the four x - terms which appear are designated by the abbreviations a_1, a_2, a_3, a_4. These values are as follows:

$$a_1 = \frac{n^2}{1-\sigma} + \alpha^2 \qquad a_2 = -(n^2-1)(n^2+\alpha^2) - \sigma\alpha^2$$
$$a_3 = -\frac{\alpha^2}{1-\sigma} \qquad a_4 = \alpha^2(n^2+\alpha^2-1) \qquad \text{-------(15)}$$

In order that these three equations (I') to (III') shall be consistent the determinant of their coefficients must vanish. (Translator's note: See "Theory of Equations", Dickson, p.121). This gives the required relation between x and y.

4. Equation for the Buckling Pressure.

The equation for the determination of the critical value of y, namely,

$$\begin{vmatrix} (1-\sigma)n^2 + 2\alpha^2 & -(n^2+\alpha^2) & \alpha^2(1-\sigma) \\ \sigma n^2 - \alpha^2 & n^2+\alpha^2+a_1 x & a_2 x \\ \sigma n^2 - \alpha^2 - \sigma & n^2+\alpha^2-1+a_3 x & a_4 x + y(1-n^2) \end{vmatrix} = 0 \qquad \text{-------- (16)}$$

has, as one can easily determine, the form

$$y(A + Bx) = C + Dx + Ex^2 \qquad \text{----------------(17)}$$

The coefficients A to E can easily be determined separately. We obtain A equal to $(1 - n^2)$ times the two-rowed determinant to the left in equation (16) above, with a_1 set equal to zero.

$$A = (1 - n^2)(n^2 + \alpha^2)\left[(1-\sigma)n^2 + 2\alpha^2 + n^2\sigma - \alpha^2\right]$$
$$= (1 - n^2)(n^2 + \alpha^2)^2$$

In the same manner, except for the factor $1 - n^2$, B is equal to a_1 times the first member of equation (16).

$$B = (1 - n^2)\left[n^2 + \frac{2\alpha^2}{1-\sigma}\right]\left[n^2 + (1-\sigma)\alpha^2\right]$$

The absolute term C of eq. (17), except for the sign, has the value of the determinant eq. (16) if x and y are set equal to zero in it.

$$-C=(1-\sigma)\,\alpha^2\begin{vmatrix}\sigma n^2-\alpha^2 & n^2+\alpha^2\\ \sigma n^2-\alpha^2-\sigma & n^2+\alpha^2-1\end{vmatrix}=\alpha^4(1-\sigma^2)$$

The coefficient E of x^2 is also found in a simple manner.

$$-E=\left[(1-\sigma)\,n^2+2\alpha^2\right]\begin{vmatrix}a_1 & a_2\\ a_3 & a_4\end{vmatrix}=\alpha^4(n^2+\alpha^2)\left[(1-\sigma)\,n^2+2\alpha^2\right]$$

The determination of D is somewhat more cumbersome, since it requires the combination of two determinants.

$$-D=\begin{vmatrix}(1-\sigma)n^2+2\alpha^2 & -(n^2+\alpha^2) & (1-\sigma)\,\alpha^2\\ 0 & a_1 & a_2\\ \sigma n^2-\alpha^2-\sigma & n^2+\alpha^2-1 & 0\end{vmatrix}$$

$$+\begin{vmatrix}(1-\sigma)\,n^2+2\alpha^2 & -(n^2+\alpha^2) & (1-\sigma)\,\alpha^2\\ \sigma n^2-\alpha^2 & n^2+\alpha^2 & 0\\ 0 & a_3 & a_4\end{vmatrix}$$

The computation leads to the value,

$$-D=(n^2+\alpha^2)^4-2n^2\left[n^2+(2+\sigma)\,\alpha^2\right]\left[n^2+\tfrac{3-\sigma}{2}\,\alpha^2\right]$$
$$+\left[n^2+(1-\sigma)\alpha^2\right]\left[n^2+2\,(1+\sigma)\,\alpha^2\right]$$

(Translator's note: This value can be readily checked if we make the substitution $n^2+\alpha^2=Z$ and collect in powers of Z.)

Thus eq. (17) is completely solved. In order to write this expression in a simpler form, we can make the substitution of the quotient

$$\frac{\alpha^2}{n^2+\alpha^2}=\varrho \qquad \text{-------(18)}$$

in place of α. After this simplification we get the following equation (A)

$$y\left[1+x(1-\sigma\varrho)\left(1+\tfrac{1+\sigma}{1-\sigma}\varrho\right)\right]=\frac{1-\sigma^2}{n^2-1}\varrho^2$$
$$+\frac{x}{n^2-1}\left[\frac{n^4}{(1-\varrho)^2}-2n^2\left\{1+(1+\sigma)\varrho\right\}\left\{1+\tfrac{1-\sigma}{2}\varrho\right\}\right.$$
$$\left.+(1-\sigma\varrho)\left\{1+(1+2\sigma)\varrho\right\}\right]+x^2\frac{n^4}{n^2-1}\,\frac{\varrho^2}{(1-\varrho)^2}\left[1-\sigma+(1+\sigma)\varrho\right]$$
$$\text{---------------------(A)}$$

(Translator's note: In the text the last term in the bracket on the left hand side is given $\frac{\sigma}{1-\sigma}\varrho$).

If we now replace σ by its value 0.3, equation (A) takes the form:

$$y\left[1+x(1-0.3\varrho)\,(1+1.86\varrho)\right]=\frac{0.91}{n^2-1}\varrho^2+\frac{x}{n^2-1}\left[\frac{n^4}{(1-\varrho)^2}-2n^2(1+1.3\varrho)\right.$$
$$\left.(1+0.35\varrho)+(1-0.3\varrho)(1+1.6\varrho)\right]+\frac{\varrho^2}{(1-\varrho)^2}\,\frac{n^4}{n^2-1}\,(0.7+1.3\varrho)\,x^2 \quad \text{-----(A')}$$

(Translator's note: In the text, the last term in the bracket on the left hand side is given 0.43ϱ)

Since y is equal to p, except for a known factor, we have in equation (A) the required expression.

5. Simplification of the Equation.

The equation (A) represents an hyperbola in an x-y coordinate system.

In the particular case $\rho = 0$ (tube of infinite length), equation (A) reduces to

$$y\,(1 + x) = x(n^2 - 1) \qquad \text{--------------------(19)}$$

From the definition in eq. (14) it follows that y represents the magnitude of the elastic compressive strain of the tube wall under the pressure p. Even though we go beyond the proportional limit of the material (see further below), y for iron or other metal will amount to not more than a few thousandths at the most. Now the horizontal asymptote of the hyperbola, eq. (19), has the ordinate $n^2 - 1$, which is at least 3. We see, therefore, that the particular part of the hyperbola in question has only a very slight curvature and can therefore be represented by a straight line with sufficient accuracy. We omit, accordingly, the last member to the right in eq. (A) and bring the value of y, considerably shortened, on the other side. (Translator's note: The omission of the x^2 term is completely justified since x is always very small and the coefficients of the x and x^2 terms are of the same order of magnitude. An actual computation for the E.M.B. model S III 125D50T1 gives $x = 2.632 \times 10^{-6}$, $x^2 = 6.927 \times 10^{-12}$, $\rho = 0.3815$, and using $n = 16$, eq. (A') reduces to

$$y(1 + 3.98 \times 10^{-6}) = 519.2 \times 10^{-6} + 668.5x + 116.9x^2$$
$$= (519.2 + 1759.2 + 0.00081) \times 10^{-6}$$

since the x^2 term contributes only 0.00081 as compared with 1759.2 for the x term it is surely negligible.). If we remove, then, the factor $(n^2 - 1)^2$ from the coefficient of x we have:

$$y = \frac{1-\sigma^2}{n^2-1}\,\rho^2 + x\left[n^2 - 1 + \frac{\lambda_1 n^4 + \lambda_2 n^2 + \lambda_3}{n^2-1}\right]$$

where

$$\lambda_1 = \frac{\rho(2-\rho)}{(1-\rho)^2}, \quad \lambda_2 = -\rho\left[3+\sigma+(1-\sigma^2)\rho\right]$$
$$\lambda_3 = \rho(1+\sigma) - \rho^2\left[\sigma(1+2\sigma)+(1-\sigma^2)(1-\sigma\rho)\left(1+\frac{1+\sigma}{1-\sigma}\rho\right)\right] \qquad \text{--------(B)}$$

(Translator's note: In the text, the denominator of λ_1 is given $1 - \rho^2$).

In general, it is sufficient to regard ρ as a small quantity and to neglect higher powers of ρ in computing λ. (Translator's note: The values of ρ for E.M.B. models S III 2000D 50T1 to S III 125D 50T1, which represent lengths of from 2D to 0.125D, vary from 0.024 to 0.33 and their squares vary from 0.0006 to 0.11. The neglect of ρ^2 terms causes an error of less than 1% for models whose frame spacing is equal to one half the diameter or greater. However, for the E.M.B. model SX 154D 50T1 with a frame spacing of 0.137D, the error is 17%. This frame spacing corresponds to that commonly used in submarine design. The slight error noted in (A) where $\frac{\sigma}{1-\sigma}$ is given in place of $\frac{\sigma+1}{1-\sigma}$ is unimportant since it is multiplied by ρ^2 in the last term of (B) and disappears.) It becomes

evident then that the coefficient of ϱ will become divisible by $n^2 - 1$ and we find in place of equation (B)

$$y = \frac{1-\sigma^2}{n^2-1}\varrho^2 + x\left[n^2 - 1 + \frac{\varrho}{1-2\varrho}(2n^2 - 1 - \sigma)\right] \qquad \text{--------(C)}$$

If, in eq. (C), we substitute the value of ϱ from eq.(18) and the values of $\underline{x}$ and $\underline{y}$ from eq. (14) and place $\sigma = 0.3$, we finally obtain the equation for the critical pressure, $\underline{p}$:

$$p = \frac{2E}{(n^2-1)\left[1+\left(\frac{nl}{\pi a}\right)^2\right]^2}\frac{h}{a} + 0.73E\left[n^2-1+\frac{2n^2-1.3}{\left(\frac{nl}{\pi a}\right)^2-1}\right]\frac{h^3}{a^3} \qquad \text{--------(D)}$$

(Translator's note: Equation (C) is given incorrectly in the text where the denominator $1 - 2\varrho$ is omitted. This mistake follows from the use of $1 - \varrho^2$ for $(1-\varrho)^2$ in the denominator of λ_1. This same error also affects the final formula D. The denominator of the last term, given as $1 + \left(\frac{nl}{\pi a}\right)^2$ in the text, actually becomes $\left(\frac{nl}{\pi a}\right)^2 - 1$ when the corrected equation (C) is used.).

If we omit the fraction that stands in the parenthesis beside $n^2 - 1$, we have the approximate solution of Southwell, which has been given above. For the general range of application, eq. (D) gives directly, as can be seen, a useful approximation. (Translator's note: Southwell's equation and eq. (D) become identical when in (D) we neglect 1 in comparison with $\left(\frac{nl}{\pi a}\right)^2$ and omit the fraction noted above, and in Southwell's equation we place Z equal to $\frac{\pi^4}{16}$. This value of Z is derived by Southwell (loc. cit.) for the ideal type of end constraints which merely keep the ends circular without imposing any other restrictions upon the types of distortion, and was also used by Cook, - Phil.Mag., Oct., 1925, pp. 844-8. The value of $\left(\frac{nl}{\pi a}\right)^2$ varies from 40 for the E.M.B. model series III 2000D50T1, to 2.8 for the E.M.B. model series III 125D50T1.)

The quantities that stand on the right hand side of eq.(D) are all given directly with the exception of $\underline{n}$: that is, the length of the tube, l; the radius, $\underline{a}$; the wall thickness, 2h; and the elastic modulus, E. Concerning $\underline{n}$, which must be a whole number, more will be said later.

6. Discussion of Results.

If we set $\varrho = 0$ in eq. (C), we obtain for the tube of infinite length,

$$y = (n^2 - 1)\,x$$

In the $\underline{x}$-$\underline{y}$-coordinate system the lines converge at the origin and those with the greater slopes have the greater values of $\underline{n}$. The smallest value other than zero is for $\underline{n} = 2$, that is, $\underline{y} = 3x$, or, if we substitute in eq. (D), $l = \infty$ and $n = 2$ we get $\underline{p} = 2.19\,E\left(\frac{h}{a}\right)^3$.

This is identical with the above-mentioned equation (a) with σ = 0.3.

If a/l and, therefore, ρ have values other than zero, equation (C) represents a particular straight line which will cut the axis of ordinates in some point for each value of n. The ordinates of the point of intersection - equal to the term which is free from x in eq. (C) - decrease with increasing n and constant a/l, while simultaneously the slopes of the lines - represented by the coefficient of x - increase. It follows, therefore, that the straight lines represent a portion of a polygon that bends downward.

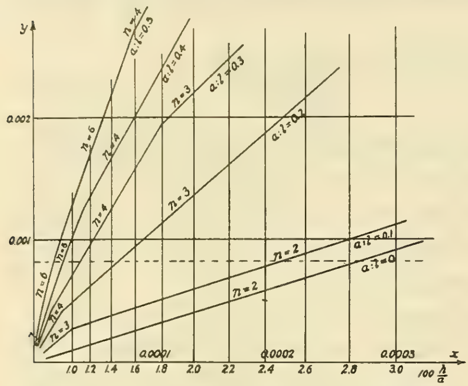
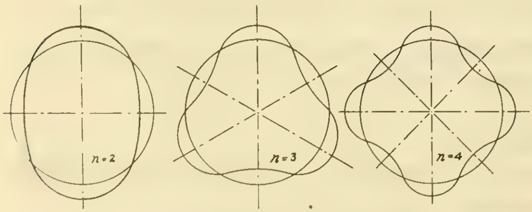


Fig. 3. The least critical pressure for various wall thicknesses 2h and tube lengths l.

The point x = 0, y = 0, is attained only for the polygon, n = ∞. Now n, however, signifies the number of lobes that appear in the circumference at the time of collapse. the figures 4 to 6 show the shape of the deformation of the circle through $w = C \sin n\varphi$ when n = 2, 3, 4. (Translator's note: w is the displacement in the radial direction.). We have, therefore, the result that the smaller the wall thickness in proportion to the radius, the greater the number of incipient waves. (Translator's note: This can be seen directly from Fig. 3). Of course, in the resulting deformation, not all the waves will appear completely formed, but each wave will have a length equal to the circumference divided by n.



Figs. 4 to 6. Deformations of the circular cross section for n equals 2 to 4.

In Fig. 3, the polygons are shown corresponding to the values of a/l = 0.5, 0.4 0.3, 0.2, .0.1, and the straight line for $l = \infty$. Fig. 7 shows the lines to four times the scale for the particular region in question to about h/a = 0.014. In both figures, the lengths along the x-axis (on which $x = \frac{1}{3}\left(\frac{h}{a}\right)^2$ is given simultaneously) are given in values of $100\frac{h}{a}$.

Table 1 contains those values of the whole number n for the above given values of a/l and for intervals equal to 0.002 of the ratio h/a, which give the least critical pressure. The table is made from direct readings from Figs. 3 and 7.

Further, Table 2 also contains the magnitudes of the least critical pressures, for the same values of a/l and h/a. Equation (D) is the basis for calculations, but improvements are taken into consideration with regard to the more exact equations (A) and (B). The values of p given in the table are valid for an elastic modulus of $E = 2{,}000{,}000\ kg/cm^2$ (28×10^6 lbs. per sq. in.) and can be changed for metals with another modulus E' in the ratio E'/E.

In both tables those values are cut off at the lower right hand side which correspond to stress of more than 1800 kg/cm^2 (25,600 lbs.per sq.in.)

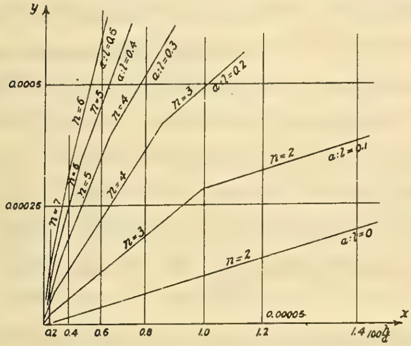


Fig. 7. Fig. 3 to a larger scale.

$$p \frac{a}{2h} \geqq 1800 \qquad \text{-------(21)}$$

In the same way, a dotted line is drawn in Fig. 3, which corresponds to eq. (21), namely

$$y = p \frac{a}{2h} \frac{1 - \sigma^2}{E}$$

$$= 1800 \frac{0.91}{2 \times 10^6} \qquad \text{-------(22)}$$

$$= 8.19 \times 10^{-4}$$

We will return to the significance of this demarcation immediately.

If the values of h, a, and l are such that the tables are not sufficient, we must determine the pressure p for various values of n from eq. (D). The determinative value is then the least of these values. In Fig. 8 are represented on the coordinates a/l and 100 h/a the limiting lines that separate, for example, the region in which n = 3 gives the smallest value of p from the region in which n = 2 or 4 is determinative. The use of Fig.8 saves many trials.

$\frac{100h}{a}$ =	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$\frac{a}{l}$ = 0	2	2	2	2	2	2	2	2
0.1	4	3	3	3	2	2	2	2
0.2	5	4	4	4	3	3	3	3
0.3	6	5	5	4	4	4	4	4
0.4	7	6	5	5	5	4	4	4
0.5	8	6	6	5	5	5	5	4

Table 1.
The number of lobes n around the circumference for various wall thicknesses 2h and tube lengths l.

$\frac{100h}{a}$ =	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$\frac{a}{l}$ = 0	0.035	0.28	0.95	2.25	4.4	7.6	12	18
0.1	0.18	1.0	2.9	0.6	12.3	17	23	31
0.2	0.37	2.1	5.9	13	21	32	47	66
0.3	0.56	3.2	9.3	18	32	51	76	111
0.4	0.76	4.5	11.6	25	45	70	101	140
0.5	0.97	5.5	15	32	55	87	132	190

Table 2.
Critical pressure p in kg. per sq. cm. for various wall thicknesses 2h and tube lengths l for $E = 2 \times 10^6$ kg. per sq. cm.

7. Comparison with Experiment.

The comparison of our results with the accumulated observations of the buckling pressure of boiler flues becomes very difficult for several reasons. First, the exact value of the elastic modulus E is almost never given for the experiments. This is of particular importance if the average pressure lies near the elastic limit, so that E sinks far below the customary value. Second, the experimental arrangements provide no certainty that the ends of the tubes can be considered as fixed, in the sense that $\underline{w} = 0$ is set for $x = \pm l/2$. Analogous to the case of ordinary buckling, we will, by not having sufficiently rigid ends, introduce a greater value of $\underline{l}$ as the effective length ("Massgebende Länge"). Finally, the given wall thicknesses are often to be regarded as the average for somewhat variable magnitudes while for the proposed problem it appears rather that the minimum value should be used.

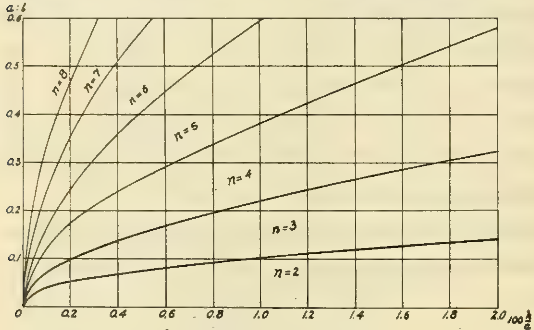


Fig. 8. Separation of the regions for the lobe numbers n equals 2 to 8.

There are to be considered: Experiments by Fairbairn (1858), by Richards (1881, discussed by Wehage, referred to by Bach), the experiments of the Danzig shipyards,(1887 to 1892) and those of A.P.Carman (1905). All those cases were eliminated for which the average stress on the surface of the material of the tube was above the proportional limit, that is, about 1800 atmospheres. (Translator's note: An atmosphere is slightly greater than a kg./cm^2).This corresponds in eq.(22) to the imposed upper limit on $\underline{y}$ of 8.19×10^{-4}. The number of experimental results to be compared, therefore is not very great.

For the Carman experiments (Physical Review 21, 1905, p.381) conducted with very small, thick-walled brass tubes, this limitation permits only the tubes of very great length to be considered: They give, as was noted by Carman himself very good agreement with the formula (a) for $l = \infty$.

For the experiments of the Danzig shipyards (Z.1894, p.689), only two lie within the valid region of our derivation, namely:

$$2a = 100 \text{ cm}, \quad l = 106.2 \text{ cm}, \quad 2h = 0.81 \text{ cm}, \quad \underline{p} = 24 \text{ atm.}$$

$2a = 100$cm, $l = 198$ cm $2h = 1.14$ cm, $p = 32$ atm.

If we set up the quotient h/a and a/l, a glance at Fig. 8 or at Table 1 shows that in the first case we must use for computation $n = 5$ and in the second $n = 4$. Our formula (D) gives then, using $E = 2 \times 10^6$kg./cm^2, $p = 30$ for the first tube and $p = 39$ for the second, which is about 23 per cent more than the observed values. In order to give an estimate of this variation, it might be remarked that for the first tube the observed value $p = 24$ would result if the wall thickness were about 0.4 mm less and the effective ("massgebende") tube length about 10% greater. For the second tube, eq. (D) would give $p = 32$ atm. if instead of 11.4 mm as wall thickness, 10.8 mm be used. These are differences which may very probably be assumed in the whole range of the experiment since it was sought to attain the most probable conditions in practice, not exact experimental conditions.

Concerning the five experiments of Richards (Engineering, 1881,I p.429. Compare Wehage, Dingler's Polyt. Jour. 1881, Vol.242, p.236), two experiments, according to the descriptions of the authors, were made on old, second-hand tubes that were lapped and riveted and therefore must have shown marked departure from the circular form. For a third experiment, the tube already had an observed bulge. For these three cases, our formula gives approximately twice as high a collapsing pressure as that actually observed. For the two other experiments (welded and strap jointed tubes) the results were exactly as for the tubes of the Danzig Navy Yard.

The experimental values are:

$2a = 96.5$ cm, $l = 218.5$ cm, $2h = 1.27$ cm, $p = 31.6$ atm;

$2a = 137.1$ cm, $l = 91.4$ cm, $2h = 0.635$ cm, $p = 9.0$ atm.

The computation by formula (D) gives in the first case $p = 45$ atm. for $n = 3$, and in the second $p = 11.9$ atm. for $n = 7$. In order to obtain the observed values from the formula, it is sufficient to increase the length about 10%, from consideration of the defective support, and decrease the wall thickness about 0.5 mm from consideration of the inequalities, the welds, etc.

The experiments of Fairbairn (Trans. Royal Society, London, 1858,p.389), which were not undertaken with actual boiler flues, but with small, carefully prepared models, furnish for the most part results which completely agree with our computations. We choose three examples at random, or, to be exact, from those which, according to Fairbairn's data, show the most uniform lobe formation.

$2a = 6$ inches, $a/l = 0.1$ $h/a = 0.00717$

p expt. = 4.6 at p calc. = 4.8 atm.

$2a = 8$ inches $a/l = 0.133$ $h/a = 0.00537$

p expt. = 2.75 at p calc. = 2.85 atm.

$2a = 8$ inches $a/l = 0.1$ $h/a = 0.00537$

p expt. = 2.18 at p calc. = 2.15 atm.

Also Fairbairn observed the increase of the number of lobes with the decrease in the length of the tube in agreement with our theory.

In general we might say that the available experimental data do not permit decisive conclusions, but point to the usefulness of the formula in design.

Concluding Remarks.

It is possible to object to the practical application of the formula here developed, since very frequently the collapse of boiler flues follows only from pressures that do not satisfy eq. (21). In these cases, the proportional limit is exceeded and our derivation is no longer valid. The behavior here is quite similar to the ordinary buckling phenomena. For rods which are not very slender, the Euler formula is no longer applicable but must be replaced with empirical formulas.

We might, however, refer here to the expedient that is applied with success in the theory of buckling. The formulas for the critical pressure remain the same if in place of E another suitable value is substituted. In the first approximation we might substitute E_1 from the slope - in general variable - of the stress-strain curve. It is more accurate, as v. Karman (Mitteilungen über Forschungsarbeiten, Vol. 81) has shown, to choose an intermediate value between the slope value and the elastic constant E. (Translator's note: The resulting modulus E' that should be used in eq. (D) is in general an intermediate value between the two moduli E - within the proportional limit - and E_1 - at failing stress - and can be expressed as $E' = \frac{4\,EE_1}{(\sqrt{E}+\sqrt{E_1})^2}$. See v. Karman, loc. cit. p.20). The place at which the slope is to be chosen must of course be found through a previous estimate.

Another range of application, for which the above derivation is again not valid can also be pointed out. For corrugated tubes the principal results can be applied. It is only necessary to introduce in the fundamental equations an increased bending rigidity in the circular section: In the expression Eq.(5) a value must be substituted for K_2 that represents the increase of the moment of

inertia calculation.

Summary.

From the elastic theory of thin, elastic shells, an accurate formula has been derived for the buckling pressure of smooth boiler flues. For the majority of cases, formula (D) is considered sufficiently accurate:

$$p = \frac{2E}{(n^2-1)\left[1+\left(\frac{nl}{\pi a}\right)^2\right]^2}\,\frac{h}{a} + 0.73E\left[n^2-1+\frac{2n^2-1.3}{\left(\frac{nl}{\pi a}\right)^2-1}\right]\frac{h^3}{a^3} \quad \text{--------(D)}$$

where, 2a is the diameter, 2h the wall thickness, l the tube length, E the elastic modulus, and where n (the number of lobes) is that whole number which makes the pressure p a minimum value. The Tables 1 and 2, and the Figures 3, 7, and 8 facilitate the application.

Comparison with the present partly imperfect experiments leads us to expect that the new formula will prove true within the proportional limit, as does the Euler formula for buckling load. Concerning the extension of the theory to non-elastic cases and to corrugated tubes, the concluding remarks above contain a suggestion as to possible procedure in regard to this problem.(Translator's note: It should be borne in mind that formula (D) differs from the formula given by von Mises in the German text and quoted by v. Sanden and Günther, (Werft und Reederei, 1920, Heft 10, p. 217) and Johow-Foerster, ("Hilfsbuch für den Schiffbau", Berlin, 1928, p. 929), where the denominator of the last term is given as $1+\left(\frac{nl}{\pi a}\right)^2$ instead of $\left(\frac{nl}{\pi a}\right)^2 - 1$. The difference is negligible for models with moderately long frame spacing, (l = a or greater), since $\left(\frac{nl}{\pi a}\right)^2$ is large compared with unity and the entire fraction is small compared with n^2. However, this difference becomes very great when attempts are made to apply the formula to the region of short frame spacings used in submarine design. The following table shows experimental and computed values for several E.M.B. models:

E.M.B. MODEL NUMBER	$\frac{L}{d}$	COLLAPSING PRESSURE, lbs.per sq.in. Experimental	Theoretical Eq.(B)	Theoretical Eq.(D)	Theoretical Incorrect Eq.(D)
SIII1000D50T1	1.0	58	51.7	51.8	51.5
SIII 500D50T1	0.5	96	100.3	101.2	97.7
SIII 250D50T1	0.25	140	232.9	243.8	214.7
SX154D50T99U1	0.137	170	425.6	497.1	336.6
SIII 125D50T1	0.125	195	406.0	479.1	312.1

E = 30,000,000 lbs. per sq. in.

Von Mises has pointed out that formula (D) is not applicable for stress-

es beyond the proportional limit unless the corrected value of E be used. It should also be noted that in this region, the values of l/a must necessarily be small. Hence ρ becomes fairly large and its square cannot be neglected. Therefore the more exact formula (B) must be used instead of the approximate formula (D) which was obtained from (B) by neglecting ρ^2. The two formulas may give values differing by 20% or more as shown in the above table.

It is evident, therefore, that formula (D) does not apply to short frame spaces. The attempts of the Germans(Johow-Foerster,"Hilfsbuch für den Schiffbau" Berlin, 1928, p.929) to apply this formula to short frame spacings by using coefficients of 0.4 to 0.6 depending upon the shell thickness were based on the assumption that the discrepancies were due to variations from circular form. It should be noted also that these coefficients were applied to the incorrect formula (D) which gives values at least 20% below those given by the exact formula (B) in this region.

If it is desired to apply these instability formulas to submarine design, the exact formula (B) should be used and also the corrected value of the elastic modulus. This latter value is, of course, very difficult to obtain. It requires an accurate knowledge of the elastic curve and the stresses at the time of collapse, since E_1 varies rapidly as the yield point of the material is approached.

On the other hand, the application of a constant multiplying factor dependent upon the shell thickness only is but a makeshift device for extracting reasonable answers from a formula in a region where it was never intended to be used and where it can not possibly give reliable results. If any multiplying factor is to be used, it should at least be a function of the stresses at the time of collapse and not of thickness only.

Tests conducted at the U.S. Experimental Model Basin show that formula (D) gives very good results for models whose length is equal to or greater than the radius, erring slightly on the side of safety. For shorter models, formula (D) gives values which are considerably too high.)

References.

(1) Bach, C., "Elastizität und Festigkeit", 1905, p. 273.

(2) Love-Timpe, "Lehrbuch der Elastizität", Leipzig, 1907, p. 637.

(3) Lorenz, R., Physikal. Zeitschrift, 1911, p. 257.

(4) Southwell, R.V., Phil. Mag., Vol.25, 1913, p. 687.

(5) Love, A.E.H., "The Mathematical Theory of Elasticity", Fourth Edition.

(6) Prescott, J., "Applied Elasticity", 1924.

(7) Dickson, "Theory of Equations".

(8) Cook, G., Phil. Mag., Oct. 1925, pp. 844-8.

(9) Carman, A.P., Phys. Review 21, 1905, p. 381.

(10) Zeitschrift 1894, p. 689. (Danziger Werft).

(11) Richards, Engineering, 1881, I., p. 429.

(12) Wehage, Dingler's Polyt. Jour., 1881, Vol.242, p. 236.

(13) Fairbairn, W., Trans. Royal Society, London 1858, p.389.

(14) Karman, T. v., Mitteilungen uber Forschungsarbeiten No.81, 1910.

(15) Johow-Foerster, "Hilfsbuch fur den Schiffbau", Berlin.

(16) v.Sanden und Gunther, Werft und Reederei, No. 10, 1920.
